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Four-wave interactions in plasmas and other nonlinear media

Frank Verheest
Seminarie voor Theoretische Mechanika, Rijksuniversiteit Gent, Krijgslaan 281-S9, B-9000 Gent, Belgium

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Abstract. The nonlinear interaction between waves in plasmas or other nonlinear media is studied using a general formalism based on two timescales and allowing for the presence of negative-energy waves when necessary. Using coupled-mode theory, based on selection rules between four frequencies and four wavenumbers, allows a description of the interaction for conservative media, where the remainder of the waves outside the selected four acts as a noise in the system through an equivalent frequency mismatch. The solutions for the interacting amplitudes are given via elliptic functions, indicating a slow oscillatory exchange of energy or an explosive instability. The mere presence of negative-energy waves is not sufficient to obtain an explosive instability and a discussion is given of the necessary conditions for such an instability. For some special cases the threshold and growth rate for the instability are calculated, and it is found that the threshold increases linearly with the total equivalent frequency mismatch.

1. Introduction

In comparison with the vast body of literature about three-wave interactions, of which an excellent survey was given by Weiland and Wilhelmsson (1977), relatively little has been published about four-wave interactions. The reason for this somewhat benign neglect lies in the fact that a four-wave interaction (through one set of selection rules) is essentially a process of an order higher than the corresponding three-wave interaction.

However, there are physically relevant situations where the selection rules for three-wave interactions cannot be fulfilled, as in the case of interaction between longitudinal plasma waves. Then a four-wave interaction becomes the first nontrivial nonlinear process. Moreover, the advent of powerful wave sources should make this effect more easily observable, especially in plasmas. Four-wave interactions are not confined to plasma physics alone, where they constitute a feature of novel interest, but have been studied before in nonlinear optics (Bloembergen 1965) or in the interaction between gravity waves on the surface of water tanks (Phillips 1974). All these interaction phenomena share some common characteristics, and it is thus of interest to give as general a description as possible, more general than the otherwise beautiful series of papers by Boyd and Turner (1972, 1973, 1977, 1978) which start from a Lagrangian description for wave–wave interactions in plasmas. In view of the basic characteristics of four-wave interactions, as found from the present paper, basic characteristics which do not depend on the detailed expressions for the coupling coefficients, it comes as no surprise that the conclusions presented here largely parallel

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those recently given for four-wave interactions in plasmas by Turner (1980). One of the salient points of our general treatment is a careful discussion of the coupling in third order between all waves present in the system. For conservative media the waves outside the selection rules for four waves can be lumped together as a kind of noise, manifesting itself as a supplementary frequency shift.

Later in the paper, special attention has been paid to the necessary and sufficient conditions for explosive instabilities. Such instabilities are a distinct possibility in plasmas where beams and negative-energy waves can easily be found, especially in astrophysical applications with its many energy sources to feed instabilities.

2. General formulation

For a given wave-interaction problem in a nonlinear medium one starts from

$$Lu = N(u), \tag{1}$$

where L is some linear and N a nonlinear operator, and u represents the unknown dependent variables. The linear approximation to u has the form of a superposition of n waves:

$$u_{\mathrm{lin}} = \sum_{j=1}^{n} a_j(t) \exp \mathrm{i}(k_j x - \omega_j t) + \mathrm{cc}. \tag{2}$$

Here a_j is the complex amplitude of the jth wave with wavevector k_j and frequency ω_j. Due to the nonlinear interaction, the amplitudes a_j could vary slowly compared to the fast phase changes of the waves. One requires u_{lin} to be a solution of the linearised form of (1),

$$L_0 u_{\mathrm{lin}} = 0, \tag{3}$$

where L_0 contains the space and fast time derivatives of L but does not take into account any slow time variation. For each wave ω_j and k_j are thus connected through a dispersion law, which can be computed once L and hence also L_0 is given. Once the linear approximation is thus known, a perturbation scheme is applied to (1), in which the second-order terms (quadratic in the wave amplitudes a_j) give rise to three-wave interactions and the third-order or cubic terms to four-wave interactions. If interactions of the three-wave type are not possible, the first significant contribution to the slow time variations of a_j will come from the four-wave interactions. Using symmetric selection rules for the first four waves (a suitable renumbering is always possible),

$$\begin{aligned} &\omega_1 + \omega_2 + \omega_3 + \omega_4 \simeq \delta, \\ &k_1 + k_2 + k_3 + k_4 = 0, \end{aligned} \tag{4}$$

with δ a small frequency mismatch, leads to a set of amplitude equations of the form

$$\begin{aligned} \frac{\partial a_1}{\partial t} &= \mathrm{i}\lambda_1 \bar{a}_2 \bar{a}_3 \bar{a}_4 \exp(-\mathrm{i}\delta t) + \mathrm{i} \sum_{l=1}^{n} \mu_{1l} a_l \bar{a}_l a_1, \\ \frac{\partial a_2}{\partial t} &= \mathrm{i}\lambda_2 \bar{a}_3 \bar{a}_4 \bar{a}_1 \exp(-\mathrm{i}\delta t) + \mathrm{i} \sum_{l=1}^{n} \mu_{2l} a_l \bar{a}_l a_2, \\ \frac{\partial a_3}{\partial t} &= \mathrm{i}\lambda_3 \bar{a}_4 \bar{a}_1 \bar{a}_2 \exp(-\mathrm{i}\delta t) + \mathrm{i} \sum_{l=1}^{n} \mu_{3l} a_l \bar{a}_l a_3, \end{aligned} \tag{5}$$

$$\frac{\partial a_4}{\partial t} = \mathrm{i}\lambda_4 \bar{a}_1 \bar{a}_2 \bar{a}_3 \exp(-\mathrm{i}\delta t) + \mathrm{i} \sum_{l=1}^{n} \mu_{4l} a_l \bar{a}_l a_4,$$

$$\frac{\partial a_j}{\partial t} = \mathrm{i} \sum_{l=1}^{n} \mu_{jl} a_l \bar{a}_l a_j \qquad (j = 5, \ldots, n).$$

The coupling coefficients are $\mathrm{i}\lambda_j$ $(j = 1, 2, 3, 4)$, insofar as the coupling is resonant, i.e. determined by the selection rules or resonance conditions (4), and $\mathrm{i}\mu_{jl}$ $(j, l = 1, \ldots, n)$ for the nonresonant part of the interaction. The bar signifies complex conjugation. It is a peculiarity of third-order compared with second-order interaction that the nonresonant terms seem to couple all the waves present in the nonlinear medium, even without specific selection rules. There are various ways of deriving (5), either through a time-averaging over the fast phase changes in the third-order equation derived upon expanding (1) or the equivalent use of a multiple-timescale formalism. The intermediate steps in going from (1) to (5) are given elsewhere (Verheest 1976, 1980), but these are not needed to follow the subsequent discussion.

(5) can be rewritten as

$$\bar{a}_j \frac{\partial a_j}{\partial t} = \mathrm{i}\lambda_j \bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4 \exp(-\mathrm{i}\delta t) + \mathrm{i} \sum_{l=1}^{n} \mu_{jl} a_l \bar{a}_l a_j \bar{a}_j \qquad (j = 1, 2, 3, 4),$$
$$\bar{a}_j \frac{\partial a_j}{\partial t} = \mathrm{i} \sum_{l=1}^{n} \mu_{jl} a_l \bar{a}_l a_j \bar{a}_j \qquad (j = 5, \ldots, n). \tag{6}$$

In general, λ_j and μ_{jl} will be functions of the different k_m and ω_m involved, and these could be computed if the starting point (1) were known explicitly. If this is not the case, one uses the general principle that (1) is invariant for a reversal of space and time, which is usually the case in the absence of dissipation. This means in (2) that

$$a_j(-t) = \bar{a}_j(t), \tag{7}$$

and this in turn shows that in (5) or (6) λ_j and μ_{jl} as defined are real quantities, which is why the imaginary unit was written separately from the outset. A more detailed discussion of the use and limitations of this principle is given by Verheest (1980).

From (6) it thus follows that

$$\partial(a_l \bar{a}_l)/\partial t = 0 \qquad (l = 5, \ldots, n). \tag{8}$$

The amplitudes of the waves outside the interacting quadruplet remain constant in modulus and the first four equations in (6) can be rewritten as:

$$\bar{a}_j \frac{\partial a_j}{\partial t} = \mathrm{i}\lambda_j \bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4 \exp(-\mathrm{i}\delta t) + \mathrm{i} \sum_{l=1}^{4} \mu_{jl} a_l \bar{a}_l a_j \bar{a}_j + \mathrm{i}\nu_j a_j \bar{a}_j, \qquad (j = 1, 2, 3, 4) \tag{9}$$

if

$$\nu_j = \sum_{l=5}^{n} \mu_{jl} a_l \bar{a}_l \tag{10}$$

is the constant influence of the noise outside the quadruplet of interacting waves. From now on the summation indices will only run from 1 to 4 unless explicitly indicated otherwise. With the substitution

$$a_j = \frac{|\lambda_j|^{1/2}}{|\lambda_1 \lambda_2 \lambda_3 \lambda_4|^{1/4}} b_j, \qquad \xi_{jl} = \frac{|\lambda_l|}{|\lambda_1 \lambda_2 \lambda_3 \lambda_4|^{1/2}} \mu_{jl}, \tag{11}$$

(9) is transformed into

$$\bar{b}_j \frac{\partial b_j}{\partial t} = \mathrm{i} s_j \bar{b}_1 \bar{b}_2 \bar{b}_3 \bar{b}_4 \exp(-\mathrm{i}\delta t) + \mathrm{i} \sum_l \xi_{jl} b_l \bar{b}_l b_j \bar{b}_j + \mathrm{i} \nu_j b_j \bar{b}_j, \tag{12}$$

where s_j is the sign of λ_j. In this way the resonant coupling coefficients are all reduced to ± 1.

Putting

$$b_j = \rho_j \exp \mathrm{i}\alpha_j \tag{13}$$

and splitting (12) into its real and imaginary parts yields

$$\begin{aligned} \frac{\partial \rho_j}{\partial t} &= \frac{s_j}{\rho_j} \rho_1 \rho_2 \rho_3 \rho_4 \sin \psi, \\ \frac{\partial \alpha_j}{\partial t} &= \frac{s_j}{\rho_j^2} \rho_1 \rho_2 \rho_3 \rho_4 \cos \psi + \sum_l \xi_{jl} \rho_l^2 + \nu_j, \end{aligned} \tag{14}$$

where ψ stands for $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \delta t$. An equation for ψ is readily found as

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \sum_j \frac{s_j}{\rho_j^2} \rho_1 \rho_3 \rho_2 \rho_4 \cos \psi + \sum_j \sum_l \xi_{lj} \rho_j^2 + \sum_j \nu_j + \delta \\ &= \sum_j \frac{s_j}{\rho_j^2} \rho_1 \rho_2 \rho_3 \rho_4 \cos \psi + \sum_j m_j \rho_j^2 + \Delta. \end{aligned} \tag{15}$$

Here m_j is

$$m_j = \sum_l \xi_{lj} = \frac{|\lambda_j|}{|\lambda_1 \lambda_2 \lambda_3 \lambda_4|^{1/2}} \sum_l \mu_{lj}, \tag{16}$$

and Δ represents the total equivalent phase mismatch, due in part to the original frequency mismatch and in part to the outside noise:

$$\Delta = \delta + \sum_j \nu_j = \delta + \sum_{j=1}^{4} \sum_{l=5}^{n} \mu_{jl} |a_l|^2, \tag{17}$$

keeping (10) in mind.

Upon inspection of (14) one sees that

$$s_1 \rho_1 (\partial \rho_1 / \partial t) = s_2 \rho_2 (\partial \rho_2 / \partial t) = s_3 \rho_3 (\partial \rho_3 / \partial t) = s_4 \rho_4 (\partial \rho_4 / \partial t), \tag{18}$$

leading after integration to a first set of invariants, the Manley-Rowe (1956) relations. Before writing them down, one can check on (14) that the case where three or all of the s_j equal -1 can be transformed into the case where three or all of the s_j equal $+1$ by shifting ψ to $\psi + \pi$. Hence in all generalisation at least two of the signs s_j can be taken to be positive. Furthermore, for the waves with positive coupling coefficients, the difference between the squares of their amplitudes remains constant. Through a judicious renumbering one can then always take ρ_1 as the smallest amplitude among the waves with positive coupling coefficients. The Manley–Rowe relations found from (18) are then:

$$\begin{aligned} &\rho_j^2(t) = s_j \rho_1^2(t) + \gamma_j \leqslant \rho_1^2(t) + \gamma_j, \\ &\gamma_j \geqslant 0, \qquad \gamma_1 = 0, \qquad s_1 = s_2 = +1. \end{aligned} \tag{19}$$

Hence, if $\rho_1(t)$ is bounded for all t, the other $\rho_j(t)$ will be bounded as well. Now if one wave has a negative coupling coefficient, say s_4 equal to -1, it follows from (19) that

$$\rho_1^2(t)+\rho_4^2(t)=\gamma_4, \tag{20}$$

and $\rho_1^2(t)$ is bounded by γ_4, hence all $\rho_j^2(t)$ are bounded. The Manley–Rowe relations thus always lead to bounded wave amplitudes, except perhaps in the case when all coupling coefficients are positive (or negative) and instabilities may occur, depending on other conditions which will be discussed further on. There is another, independent invariant connected with the phases:

$$2\rho_1\rho_2\rho_3\rho_4\cos\psi+A\rho_1^4+B\rho_1^2+C=0, \tag{21}$$

where

$$\begin{aligned} A &= \tfrac{1}{2}\sum_j s_j m_j = \frac{1}{2|\lambda_1\lambda_2\lambda_3\lambda_4|^{1/2}}\sum_j\sum_l \lambda_j\mu_{lj}, \\ B &= \sum_j m_j\gamma_j+\Delta = \sum_{j=1}^{4}\sum_{l=1}^{n}\mu_{jl}|a_l(0)|^2-2A\rho_1^2(0)+\delta, \\ C &= -A\rho_1^4(0)-B\rho_1^2(0), \end{aligned} \tag{22}$$

if the initial phases are arranged so that $\psi(0)$ equals $\pm\frac{1}{2}\pi$. A is determined solely by the coupling coefficients, hence by the structure of the nonlinear medium, whereas B and C contain both the coupling coefficients and the initial conditions.

With the help of the Manley–Rowe relations (19) and the phase invariant (21), it is possible to eliminate from (14) and (15) all variables except one, e.g. $\rho_1(t)$, and get

$$(\partial\rho_1^2/\partial t)^2=f(\rho_1^2). \tag{23}$$

Here f is a quartic in ρ_1^2,

$$\begin{aligned} f(\rho_1^2) &= 4\rho_1^2(\rho_1^2+\gamma_2)(s_3\rho_1^2+\gamma_3)(s_4\rho_1^2+\gamma_4)-(A\rho_1^4+B\rho_1^2+C)^2 \\ &= c_0\rho_1^8+4c_1\rho_1^6+6c_2\rho_1^4+4c_3\rho_1^2+c_4, \end{aligned} \tag{24}$$

with coefficients

$$\begin{aligned} c_0 &= 4s_3s_4-A^2, \\ c_1 &= s_3s_4\gamma_2+s_4\gamma_3+s_3\gamma_4-\tfrac{1}{2}AB, \\ c_2 &= \tfrac{2}{3}(s_4\gamma_2\gamma_3+\gamma_3\gamma_4+s_3\gamma_4\gamma_2)-\tfrac{1}{3}AC-\tfrac{1}{6}B^2, \\ c_3 &= \gamma_2\gamma_3\gamma_4-\tfrac{1}{2}BC, \\ c_4 &= -C^2. \end{aligned} \tag{25}$$

Just because in (24) $f(\rho_1^2)$ is a quartic, ρ_1^2 will be given as a function of t by an elliptic function. The character of this solution will be determined solely by the location of $\rho_1^2(0)$ compared to the roots of $f(\rho_1^2)$ (Pars 1965). Hence the problem is essentially reduced to finding the roots of $f(\rho_1^2)$. On physical grounds both $f(\rho_1^2)$ and ρ_1^2 are required to be positive. If $\rho_1^2(0)$ lies between two positive and single roots of $f(\rho_1^2)$ then $\rho_1^2(t)$ will oscillate periodically between these two roots and thus be bounded. On the other hand, if $\rho_1^2(0)$ lies beyond the largest positive root (supposed simple for the time being), $\rho_1^2(t)$ is no longer bounded and an explosive instability will occur, as $\rho_1^2(t)$ reaches an infinite value in a finite time (Gradshteyn and Ryzhik 1965).

When the endpoint of the interval, in which $\rho_1^2(0)$ lies, is a multiple root of $f(\rho_1^2)$, then $\rho_1^2(t)$ will need an infinite time to approach the value of this root, and a limiting or saturation process occurs. Finally, when the largest positive root is a multiple one, and $\rho_1^2(0)$ lies beyond it, the ultimate behaviour of $\rho_1^2(t)$ depends upon the sign of $\rho_1^2(0)/\partial t$. When this is negative, $\rho_1^2(t)$ will decrease from $\rho_1^2(0)$ towards the value of the multiple root and requires an infinite time to do so. However, if $\rho_1^2(t)$ increases away from $\rho_1^2(0)$, an explosive instability occurs. Some of these cases are illustrated in figure 1.

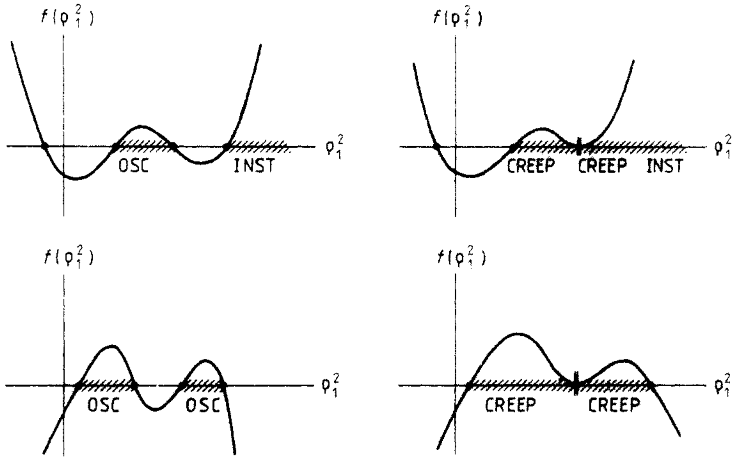

Figure 1. Examples of the curves of $f(\rho_1^2)$ against ρ_1^2, indicating regimes of oscillatory, creep or explosively unstable behaviour for ρ_1^2.

3. Explosive instabilities

Having discussed the different types of behaviour one may expect for $\rho_1^2(t)$, we will try to elucidate in this section precisely when an explosive instability may occur.

A first and necessary criterion comes from the Manley–Rowe relations (19) which, as said already, always yield bounded solutions, except when all coupling coefficients have the same sign (here positive by choice). Hence we take s_3 and s_4 equal to one. Formulated as such, this is not a very illuminating criterion, and we here need a discussion of the wave energies involved. As we have chosen to start from a symmetric selection rule (4), of necessity at least one and in all practical cases two of the wave frequencies, as considered up to now, will be negative. The total energy in the four interacting waves can be computed from the Manley–Rowe relations (19) as being $\Sigma_j s_j\omega_j\rho_j^2(t)$ and is constant, up to a slow variation due to the original phase mismatch δ. The sign of the individual wave energies is thus given by the sign of $s_j\omega_j$. In the case we are discussing, with one or two ω_j negative and all s_j positive, one or two waves will be negative-energy waves. It is worth contrasting this with the case of three-wave interaction (Coppi *et al* 1969) where the mere presence of a negative-energy wave was not only necessary but also sufficient to bring about an explosive instability. Here we see that the presence of negative-energy waves is a prerequisite for the occurrence of an explosive instability, but by no means sufficient, as discussed further on.

The next criterion is given by the sign of c_0, the leading coefficient in $f(\rho_1^2)$. If c_0 were negative, $f(\rho_1^2)$ would also become negative for large enough values of ρ_1^2, and hence only bounded solutions for ρ_1^2 are possible. The case when c_0 vanishes can be left out of the discussion here, as $f(\rho_1^2)$ then reduces to a cubic, which is the generic form encountered in *three-wave* interactions, about which many papers have been written (see e.g. Weiland and Wilhelmsson 1977). So a further necessary condition for instability is that c_0 be positive, or more explicitly:

$$-4<\sum_j m_j<4. \tag{26}$$

In contrast with the opinion advanced by Turner (1980), this is not yet sufficient to conclude that an instability will occur, because the number of positive roots of $f(\rho_1^2)$ could be one or three. Once c_0 is supposed positive, one finds that

$$\lim_{\rho_1^2\to\pm\infty} f(\rho_1^2)\to+\infty, \qquad f(0)\leqslant 0, \qquad f(\rho_1^2(0))\geqslant 0 \tag{27}$$

and thus $f(\rho_1^2)$ has at least one root in $]-\infty, 0]$ and one in $[0, +\infty[$. The case of $f(0)$ vanishing can be included as a special case. If there is only one positive root, one can immediately conclude that an instability is going to occur, whereas in the case of three positive roots, $\rho_1^2(0)$ has to be greater than the largest of these roots, which thus amounts to a threshold for the instability. However, great care has to be exercised in discussing these matters, as the determination of the roots of $f(\rho_1^2)$ and of a possible threshold is not independent of initial conditions, because both $\rho_1^2(0)$ and the γ_j enter into $f(\rho_1^2)$.

The domain where $f(\rho_1^2)$ has only one positive root is indicated in figure 2 for different values of c_1, c_2 and c_3 ($c_0>0$ and c_4 is supposed negative), using Descartes' theorem (see e.g. Mishina and Proskuryakov 1965). In the shaded areas of figure 2, $f(\rho_1^2)$ has one positive root if its discriminant (Dehn 1930)

$$D=(256/c_0^6)(g_2^3-27g_3^2) \tag{28}$$

is negative, where

$$g_2=c_0c_4-4c_1c_3+3c_2^2,$$
$$g_3=\begin{vmatrix} c_0 & c_1 & c_2\\ c_1 & c_2 & c_3\\ c_2 & c_3 & c_4 \end{vmatrix}, \tag{29}$$

and three positive roots when D is positive (see e.g. Mishina and Proskuryakov 1965).

The possibility of an instability is thus best summed up in a kind of flow chart, as shown in figure 3. It is seen that the set of conditions, requiring that all s_j be $+1$, that c_0

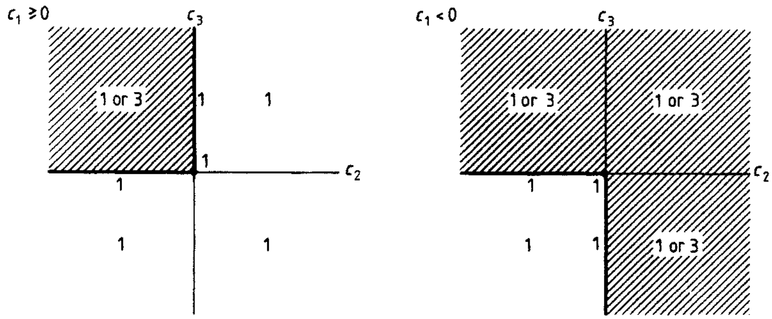

Figure 2. Regions in c_2, c_3 parameter space where $f(\rho_1^2)$ has one or three positive real roots.

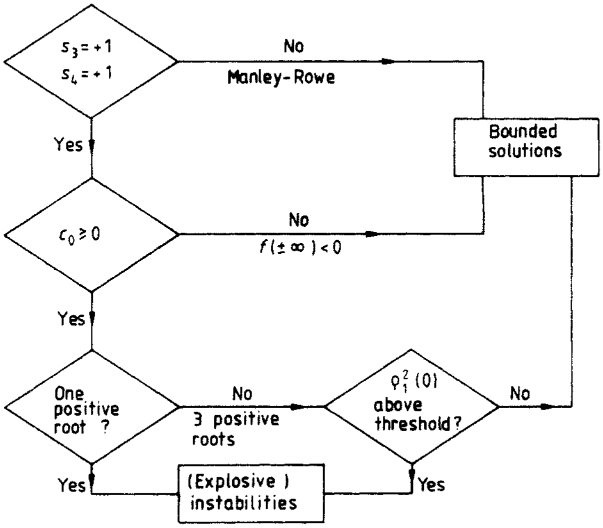

Figure 3. Chart showing the different necessary conditions in order to obtain an explosive instability.

be positive and that $\rho_1^2(0)$ be greater than the largest root of $f(\rho_1^2)$, together constitute a necessary and sufficient condition to obtain an explosive instability.

4. Special cases

As an example we consider the special case where all the waves have initially the same normalised amplitudes,

$$\rho_1^2(0)=\rho_2^2(0)=\rho_3^2(0)=\rho_4^2(0), \tag{30}$$

all the γ_j are zero and (24) reduces to

$$\begin{aligned} f(\rho_1^2) &= 4\rho_1^8-(A\rho_1^4+B\rho_1^2-A\rho_1^4(0)-B\rho_1^2(0))^2 \\ &= 4\rho_1^8-(|A|\rho_1^4+\hat{B}\rho_1^2-|A|\rho_1^4(0)-\hat{B}\rho_1^2(0))^2, \end{aligned} \tag{31}$$

where

$$\hat{B}=B\operatorname{sgn}A=\Delta\operatorname{sgn}A. \tag{32}$$

The roots of (31) are given by

$$(\rho_1^2)_{1,2}=\frac{\hat{B}\pm\sqrt{M}}{2(2-|A|)}, \qquad (\rho_1^2)_{3,4}=\frac{-\hat{B}\pm\sqrt{N}}{2(2+|A|)}, \tag{33}$$

with

$$\begin{aligned} M&=\hat{B}^2-4(2-|A|)(|A|\rho_1^4(0)+\hat{B}\rho_1^2(0)), \\ N&=\hat{B}^2+4(2+|A|)(|A|\rho_1^4(0)+\hat{B}\rho_1^2(0)). \end{aligned} \tag{34}$$

Whether these roots are real or not depends upon the signs of M and N.

When $\hat{B}$ is positive, $(\rho_1^2)_{3,4}$ are always real, whereas $(\rho_1^2)_{1,2}$ are only real as long as M is not negative. M is negative when

$$\frac{|\Delta|}{2|A|}\left[-1+\left(\frac{2}{2-|A|}\right)^{\frac{1}{2}}\right]<\rho_1^2(0), \qquad (\operatorname{sgn} A\Delta>0) \tag{35}$$

and one has the following ordering:

$$\begin{aligned} &\text{(i)} && M<0:(\rho_1^2)_3\leq 0\leq(\rho_1^2)_4<\rho_1^2(0),\\ &\text{(ii)} && M\geq 0:(\rho_1^2)_3\leq 0\leq(\rho_1^2)_4<\rho_1^2(0)<(\rho_1^2)_1\leq(\rho_1^2)_2. \end{aligned} \tag{36}$$

An explosive instability is only possible and then also occurs, when $\rho_1^2(0)$ satisfies the threshold condition (35). Otherwise all solutions are bounded. The discussion is somewhat more involved when $\hat{B}$ is negative, because then the following regimes exist for progressively larger values of $\rho_1^2(0)$:

$$\begin{aligned} &\text{(i)} && M, N\geq 0: && (\rho_1^2)_1\leq 0\leq(\rho_1^2)_2<\rho_1^2(0)<(\rho_1^2)_3\leq(\rho_1^2)_4,\\ &\text{(ii)} && M\geq 0, N<0: && (\rho_1^2)_1\leq 0\leq(\rho_1^2)_2<\rho_1^2(0),\\ &\text{(iii)} && M, N\geq 0: && (\rho_1^2)_1\leq 0\leq(\rho_1^2)_2\leq(\rho_1^2)_3\leq(\rho_1^2)_4<\rho_1^2(0),\\ &\text{(iv)} && M<0, N\geq 0: && (\rho_1^2)_3\leq(\rho_1^2)_4<\rho_1^2(0). \end{aligned} \tag{37}$$

Explosive instabilities occur in all regimes but (i), to which the threshold condition corresponds

$$\frac{|\Delta|}{2|A|}\left[1-\left(\frac{2}{2+|A|}\right)^{\frac{1}{2}}\right]<\rho_1^2(0) \qquad (\operatorname{sgn} A\Delta<0). \tag{38}$$

These thresholds differ quite a bit from those given by Turner (1980), because nothing special was supposed about the relative values of A and $B=\Delta$. The total equivalent mismatch can have either sign depending on the precise balance of the noise versus the initial frequency mismatch. The frequency mismatch δ is of necessity a small quantity, because otherwise the whole expansion scheme would become void, but the noise outside the interacting wave quadruplet need not be small in the same sense. In any case, the instability threshold, whether given by (35) or by (38), depends linearly upon $|\Delta|$. When Δ is supposed negligible,

$$(\rho_1^2)_4=[|A|/(2+|A|)]^{1/2}\rho_1^2(0) \tag{39}$$

and the instability time can be computed as

$$\tau_{\exp}=\int_{\rho_1^2(0)}^{\infty}\frac{du}{\sqrt{f(u)}}=\frac{1}{2\rho_1^2(0)\sqrt{|A|}}F\left[\sin^{-1}\left(\frac{2|A|}{2+|A|}\right)^{\frac{1}{2}},\frac{1}{2}\sqrt{2+|A|}\right], \tag{40}$$

(Gradshteyn and Ryzhik 1965) where $F(\xi, s)$ is the Legendre elliptic function of the first kind with argument ξ and modulus s. For a given nonlinear medium, A is a known quantity independent of initial conditions, and hence the product $\tau_{\exp}\rho_1^2(0)$ is constant. For a further discussion of the growth rates for explosive instabilities, the reader is referred to Turner (1980), where it is argued that such growth rates are smaller for four-wave interactions than for the better known three-wave interactions. Hence the four-wave explosive instability will be a much slower process, not only because it occurs on an intrinsically slower timescale, but also because of smaller growth rates.

However, the interest of explosive instabilities lies in the fact that one starts from three or four linearly stable waves, which slowly interact to produce an unstable situation, so that the whole picture remains stable for a rather long time (compared with a period of oscillation) before shooting up in a sudden and irrevocable way, as may be observed in some experimental or astrophysical cases such as solar flares for example.

As a general summary one can say that in the previous sections a qualitative discussion has been given of four-wave interactions in general media, with a special emphasis on a correct delimitation of when an explosive instability might occur.

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